

NON-EXISTENCE OF UNIVERSAL MEMBERS
IN CLASSES OF ABELIAN GROUPS
SH622

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ABSTRACT. We prove that if $\mu^+ < \lambda = \text{cf}(\lambda) < \mu^{\aleph_0}$, then there is no universal reduced torsion free abelian group. Similarly if $\aleph_0 < \lambda < 2^{\aleph_0}$. We also prove that if $2^{\aleph_0} < \mu^+ < \lambda = \text{cf}(\lambda) < \mu^{\aleph_0}$, then there is no universal reduced separable abelian p -group in λ . (Note: both results fail if (a) $\lambda = \lambda^{\aleph_0}$ or if (b) λ is strong limit, $\text{cf}(\mu) = \aleph_0 < \mu$).

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§0 INTRODUCTION

We deal with the problem of the existence of a universal member in \mathfrak{K}_λ . For \mathfrak{K} a class of abelian groups, \mathfrak{K}_λ is the class of $G \in \mathfrak{K}$ of cardinality λ ; universal means every other member can be embedded into it. We are concerned mainly with the class of reduced torsion free groups. Generally, on the history of the existence of universal members see Kojman-Shelah [KjSh 409]. From previous works, a natural division of the possible cardinals for such problems is:

Case 0: $\lambda = \aleph_0$. Case 1: $\lambda = \aleph_0$. Case 2: $\aleph_0 < \lambda < 2^{\aleph_0}$ Case 3: $2^{\aleph_0} + \mu^+ < \lambda = \text{cf}(\lambda) < \mu^{\aleph_0}$. Case 4: $2^{\aleph_0} + \mu^+ + \text{cf}(\lambda) < \lambda < \mu^{\aleph_0}$. Case 5: $\lambda = \mu^+$, $\text{cf}(\mu) = \aleph_0$, $(\forall \chi < \mu)(\chi^{\aleph_0} < \mu)$. Case 6: $\text{cf}(\lambda) = \aleph_0$, $(\forall \chi < \lambda)(\chi^{\aleph_0} < \lambda)$. Subcase 6a: λ is strong limit, $\text{cf}(\lambda) = \aleph_0$. Subcase 6b: Case 6 but not 6a.

Our main interest will be in Case 3 for $\mathfrak{K} = \mathfrak{K}^{\text{rtf}}$, the class of torsion free reduced abelian groups. Note that divisible torsion free abelian groups of cardinality λ are universal. A second class is $\mathfrak{K}^{\text{rs}(p)}$, the class of reduced separable p -groups (see Definition 2.3(4), more in Fuchs [Fu]). Kojman-Shelah [KjSh 455] show that for $\mathfrak{K} = \mathfrak{K}^{\text{rtf}}, \mathfrak{K}^{\text{rs}(p)}$ in Case 3 there is no universal member if we restrict the possible embeddings to pure embeddings. This stresses that universality depends not only on the class of structures but also on the kind of embeddings. In [Sh 456] we allow any embeddings, but restrict the class of abelian groups to $(< \lambda)$ -stable ones. In [Sh 552, §1, §5] we allow any embedding and all $G \in \mathfrak{K}_\lambda$ but there is a further restriction on λ related to the pcf theory. This restriction is weak in the following sense: it is not clear if there is any cardinal (in any possible universe of set theory) not satisfying it. We here prove the full theorem for $\lambda > \beth_\omega$ with no further restrictions:

(*) for $\lambda > \beth_\omega$ in Case 3, $\mathfrak{K} = \mathfrak{K}^{\text{rtf}}, \mathfrak{K}^{\text{rs}(p)}$ there is no universal member in \mathfrak{K}_λ

(where we define inductively $\beth_0 = \aleph_0, \beth_{n+1} = 2^{\beth_n}, \beth_\omega = \sum 2^{\beth_n}$ and generally $\beth_\alpha = \aleph_0 + \sum_{\beta < \alpha} 2^{\beth_\beta}$).

§1 deals with $\mathfrak{K}^{\text{rtf}}$ using mainly type theory. In §2, we apply combinatorial ideals whose definition has some built-in algebra and purely combinatorial ones to get results on $\mathfrak{K}^{\text{rs}(p)}$; there is more interaction between algebra and combinatorics than in [Sh 552]. Similarly in §3 we work on the class of \aleph_1 -free abelian groups.

What about the other cases? Case 4 (like 3 but λ singular) for $\mathfrak{K}_\lambda^{\text{rtf}}$ and pure embedding, was solved showing non-existence of universals in [KjSh 455] if some weak pcf assumption holds and in [Sh 552] this was done for embeddings under slightly stronger pcf assumptions. For both assumptions, it is not clear if they may fail. Note that the results on consistency of existence of universals in this case cannot be attacked as long as more basic pcf problems remain open.

Concerning Case 5 - If we try to prove the consistency of the existence of universals, it is natural first to prove the existence of the relevant club guessing; here we expect consistency results. (Of course, consistently there is club guessing (by $\bar{C} = \langle C_\delta : \delta \in S \rangle, S \subseteq \lambda, \text{otp}(C_\delta) = \mu$) and then there is no universal.) Also we were first of all interested in the universal reduced torsion free groups under

embeddings, but later we also looked into some of the other cases here. See more in [DjSh].

? DjSh ?

Case 1 ($\lambda = \aleph_0$). By subsequent work there is a universal member of $\mathfrak{K}_\lambda^{\text{rtf}}$, and (see Fuchs [Fu]) in $\mathfrak{K}_\lambda^{\text{rs(p)}}$ there is a universal member, but in $\mathfrak{K}_\lambda^{\aleph_1\text{-free}}$ there is no universal member (see forthcoming work).

Case 0 ($\lambda = \aleph_0$). In $\mathfrak{K}_\lambda^{\text{rtf}}$ there is no universal member (see above) and in $\mathfrak{K}_\lambda^{\text{rs(p)}}$ there is a universal member (see Fuchs [Fu]).

Case 2 ($\aleph_0 < \lambda < 2^{\aleph_0}$). For $\mathfrak{K}_\lambda^{\text{rtf}}$ we prove here that there is no universal member (by 1.2), whereas for $\mathfrak{K}_\lambda^{\text{rs(p)}}$ this is consistent with and independent of ZFC (see [Sh 550, §4]).

We also deal with Case 6 ($(\forall \chi < \lambda) \chi^{\aleph_0} < \lambda, \lambda > \text{cf}(\lambda) = \aleph_0$). There is a universal member for $\mathfrak{K}_\lambda^{\text{rtf}}$ and also for $\mathfrak{K}_\lambda^{\text{rs(p)}}$. See [DjSh].

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Notation: The cardinality of a set A is $|A|$, the cardinality of a structure G is $\|G\|$. $\mathcal{H}(\lambda^+)$ is the set of sets whose transitive closure has cardinality $\leq \lambda$ and $<_{\lambda^+}^*$ denotes a canonical well order of $\mathcal{H}(\lambda^+)$.

For an ideal I , we use I^+ to denote the family of subsets of $\text{Dom}(I)$ which are not in I .

§1 NON-EXISTENCE OF UNIVERSALS AMONG REDUCED TORSION FREE ABELIAN GROUPS

The first result (1.2) deals with λ satisfying $\aleph_0 < \lambda < 2^{\aleph_0}$ and show the non-existence of universal members in $\mathfrak{R}_\lambda^{\text{rtf}}$ which improves [Sh 552]. The proof is straightforward by analyzing subgroups and comparing Bauer's types.

Then we deal with $2^{\aleph_0} + \mu^+ < \lambda = \text{cf}(\lambda) < \mu^{\aleph_0}$. We add witnesses to bar the way against “undesirable” extensions (see [DjSh] on classes of modules) which is a ? DjSh ? critical new point compared to [Sh 552].

1.1 Definition. Let $\mathfrak{R}^{\text{rtf}}$ denote the class of torsion free reduced abelian groups G where torsion free means $nx = 0, n \in \mathbb{Z}, x \in G \Rightarrow n = 0 \vee x = 0$ and reduced means $(\mathbb{Q}, +)$ cannot be embedded into G . The subclass of $G \in \mathfrak{R}^{\text{rtf}}$ of cardinality λ is denoted by $\mathfrak{R}_\lambda^{\text{rtf}}$. Moreover, \mathfrak{R}^{tf} is the class of torsion free abelian groups.

1.2 Claim. 1) If $\aleph_0 < \lambda < 2^{\aleph_0}$ then $\mathfrak{R}_\lambda^{\text{rtf}}$ has no universal member.
2) Moreover, there is no member of $\mathfrak{R}_\lambda^{\text{rtf}}$ universal for $\mathfrak{R}_{\aleph_1}^{\text{rtf}}$.

Proof. Let \mathbf{P}^* be the set of all primes and let $\{\mathbf{Q}_i : i < 2^{\aleph_0}\}$ be a family of infinite subsets of \mathbf{P}^* , pairwise with finite intersection. Let $\rho_\alpha \in {}^\omega 2$ for $\alpha < \omega_1$ be pairwise distinct. Let H^* be the divisible torsion free abelian group with $\{x_\alpha : \alpha < \omega_1\}$ a maximal independent subset. For $i < 2^{\aleph_0}$ let H_i^* be the subgroup of H^* generated by

$$\begin{aligned} & \{x_\alpha : \alpha < \omega_1\} \cup \{p^{-n}x_\alpha : p \in \mathbf{P}^* \setminus \mathbf{Q}_\alpha, \alpha < \omega_1 \text{ and } n < \omega\} \\ & \cup \{p^{-n}(x_\alpha - x_\beta) : \alpha, \beta < \omega_1 \text{ and } p \in \mathbf{P}^* \text{ and} \\ & \quad \rho_\alpha \upharpoonright p = \rho_\beta \upharpoonright p \text{ and } n < \omega\}. \end{aligned}$$

Clearly $H_i^* \in \mathfrak{R}^{\text{rtf}}$ and $\|H_i^*\| = \aleph_1 \leq \lambda$. Let $G \in \mathfrak{R}_\lambda^{\text{rtf}}$, and we shall prove that at most λ of the groups H_i^* are embeddable into G .

So assume $Y \subseteq 2^{\aleph_0}$, $|Y| > \lambda$ and for $i \in Y$ we have h_i an embedding of H_i^* into G and we shall derive that G is not reduced; a contradiction. We choose by induction on n a set $\Gamma_n \subseteq {}^n \lambda$ and pure abelian subgroups G_η of G for $\eta \in \Gamma_n$, as follows. For $n = 0$ we let $\Gamma_0 = \{<>\}$ and let $G_{<>} = G$. For $n + 1$, for $\eta \in \Gamma_n$ such that $\|G_\eta\| > \aleph_0$ we let $\Gamma_{n,\eta} = \{\eta \hat{\ } \langle \zeta \rangle : \zeta < \|G_\eta\|\}$, and let $\bar{G}_\eta = \langle G_{\eta \hat{\ } \langle \zeta \rangle} : \zeta < \|G_\eta\| \rangle$ be an increasing continuous sequence of subgroups of G_η of cardinality $< \|G_\eta\|$ with union G_η such that:

- (*) for $\zeta < \|G_\eta\|$ we have
 $G_{\eta \hat{\ } \langle \zeta \rangle} = G_\eta \cap (\text{the Skolem Hull of } G_{\eta \hat{\ } \langle \zeta \rangle} \text{ in } (\mathcal{H}(\lambda^+), \in, <_{\lambda^+}^*, G_\eta)).$

Let $\Gamma_{n+1} = \{\eta \hat{\ } \langle \zeta \rangle : \eta \in \Gamma_n, \|G_\eta\| > \aleph_0 \text{ with } \zeta < \|G_\eta\|\}$ and $\Gamma = \bigcup_{n < \omega} \Gamma_n$.

For each $i \in Y$, let $\eta = \eta_i \in \Gamma$ be such that:

- (a) $\{\alpha < \omega_1 : h_i(x_\alpha) \in G_{\eta_i}\}$ is uncountable
- (b) under (a), the cardinality of G_{η_i} is minimal.

Clearly η_i is well defined as (a) holds for $\eta = \langle \rangle$ and clearly G_{η_i} is uncountable. Let $X_i = \{\alpha < \omega_1 : h_i(x_\alpha) \in G_{\eta_i}\}$, and let $\beta_i < \omega_1$ be minimal such that $\{\rho_\alpha : \alpha \in \beta_i \cap X_i\}$ is a dense subset of $\{\rho_\alpha : \alpha \in X_i\}$. Let $\zeta_i < \|G_{\eta_i}\|$ be the minimal ζ such that $\{h_i(x_\alpha) : \alpha \in \beta_i \cap X_i\} \subseteq G_{\eta \smallfrown \langle \zeta \rangle}$. Now the set $X'_i = \{\alpha < \omega_1 : h_i(x_\alpha) \in G_{\eta_i \smallfrown \langle \zeta_i \rangle}\}$ is countable, and hence we can find $\alpha_i \in X_i \setminus X'_i$.

Now the number of possible sequences $\langle \eta_i, \beta_i, \zeta_i, \alpha_i, h_i(x_{\alpha_i}) \rangle$ is at most $|\omega^\omega| \times \aleph_1 \times \lambda \times \aleph_1 \times \lambda$ (as $\Gamma \subseteq \omega^{>\lambda}$). So for some $\langle \eta, \beta, \zeta, \alpha, y \rangle$ and $i_0 < i_1$ from Y we have (for $\ell = 0, 1$)

$$\eta_{i_\ell} = \eta, \beta_{i_\ell} = \beta, \zeta_{i_\ell} = \zeta, \alpha_{i_\ell} = \alpha, h_{i_\ell}(x_{\alpha_\ell}) = y.$$

Now as h_{i_ℓ} embeds $H_{i_\ell}^*$ into G and $h_{i_\ell}(x_\alpha) = y$ necessarily

(*) if $p \in \mathbf{P}^* \setminus \mathbf{Q}_{i_\ell}$ and $n < \omega$ then in G, p^{-n} divides y .

So this holds for every $p \in (\mathbf{P}^* \setminus \mathbf{Q}_{i_0}) \cup (\mathbf{P}^* \setminus \mathbf{Q}_{i_1}) = \mathbf{P}^* \setminus (\mathbf{Q}_{i_0} \cap \mathbf{Q}_{i_1})$.

Now $\mathbf{Q}_{i_0} \cap \mathbf{Q}_{i_1}$ is finite so let $p^* \in \mathbf{P}^*$ be above its supremum. As $\{\gamma : \gamma \in X'_{i_0}\}$ is a dense subset of $\{\rho_\alpha : \alpha \in X_{i_0}\}$, there is $\gamma \in X'_{i_0}$ such that $\rho_\gamma \upharpoonright p^* = \rho_\alpha \upharpoonright p^* (= \rho_{\alpha_{i_0}} \upharpoonright p^*)$. Let $h_{i_0}(x_\gamma) = y^*$, it is in $G_{\eta \smallfrown \langle \zeta \rangle}$.

So in $(\mathcal{H}(\lambda^+), \in, <_{\lambda^+}^*, G_\eta)$, the following formula is satisfied

$$\begin{aligned} \varphi(y, y^*) = & \text{“in } G_\eta, y \text{ is divisible by } p^n \text{ when } p \in \mathbf{P}^* \text{ \& } p \geq p^* \text{ \& } n < \omega \\ & \text{and } y - y^* \text{ is divisible by } p^n \text{ when} \\ & p \in \mathbf{P}^* \text{ \& } p < p^* \text{ \& } n < \omega\text{”}. \end{aligned}$$

Hence by (*), i.e. by the choice of $\langle G_{\eta \smallfrown \langle \xi \rangle} : \xi < \|G_\eta\| \rangle$ necessarily for some $y' \in G_{\eta \smallfrown \langle \zeta \rangle}$ we have $\varphi(y', y^*)$. Now $y \neq y'$ as $y' \in G_{\eta \smallfrown \langle \zeta \rangle}, y \notin G_{\eta \smallfrown \langle \zeta \rangle}$. Also $y - y'$ is divisible by p^n for $p \in \mathbf{P}^*, n < \omega$.

[Why? If $p \geq p^*$ because both y and y' are divisible by p^n and if $p < p^*$ because $y - y' = (y - y^*) - (y' - y^*)$ and both $y - y^*$ and $y' - y^*$ are divisible by p^n .]

As G is torsion free, the pure closure in G of $\langle \{y - y'\} \rangle_G$ is isomorphic to \mathbb{Q} , a contradiction to “ G is reduced”. $\square_{1.2}$

1.3 Definition. 1) Let \mathbf{P}^* be the set of primes.

2) For $G \in \mathfrak{K}^{\text{rtf}}$ and $x \in G \setminus \{0\}$ let

- (a) $\mathbf{P}(x, G) = \{p \in \mathbf{P}^* : x \in \bigcap_{n < \omega} p^n G,$
equivalently x is divisible by p^n
in G for every $n < \omega\}$
- (b) $\mathbf{P}^-(x, G) = \{p : p \in \mathbf{P}^*, \text{ but } p \notin \mathbf{P}(x, G)$
and there is $y \in G \setminus \{0\}$ such that
 $\mathbf{P}^* \setminus \{p\} \subseteq \mathbf{P}(y, G) \text{ and } p \in \mathbf{P}(x - y, G)\}$.

- 3) $G \in \mathfrak{K}^{\text{rtf}}$ is called full if: for every $x \in G \setminus \{0\}$ we have $\mathbf{P}^* = \mathbf{P}(x, G) \cup \mathbf{P}^-(x, G)$.
 4) The class of full $G \in \mathfrak{K}^{\text{rtf}}$ is called $\mathfrak{K}^{\text{stf}}$ and $\mathfrak{K}_\lambda^{\text{stf}} = \mathfrak{K}^{\text{stf}} \cap \mathfrak{K}_\lambda^{\text{rtf}}$, (why s ? as the successor of r in the alphabet).

1.4 Fact. 1) If $G \in \mathfrak{K}^{\text{rtf}}$, then for any $x \in G$ the sets $\mathbf{P}(x, G)$ and $\mathbf{P}^-(x, G)$ are disjoint subsets of \mathbf{P}^* .

2) If G_2 is an extension of G_1 , both in $\mathfrak{K}^{\text{rtf}}$ and $x \in G_1 \setminus \{0\}$ then

- (a) $\mathbf{P}(x, G_1) \subseteq \mathbf{P}(x, G_2)$, with equality if G_1 is a pure subgroup of G_2
- (b) $\mathbf{P}^-(x, G_1) \subseteq \mathbf{P}^-(x, G_2)$.

3) For every $G \in \mathfrak{K}^{\text{rtf}}$ there is a G' such that

- (a) G' is full, $G' \in \mathfrak{K}^{\text{rtf}}$
- (b) G is a pure subgroup of G' , $\|G'\| = \|G\|$.

Proof. 1), 2) Trivial.

3) It suffices to show

- (*) if $G \in \mathfrak{K}^{\text{rtf}}$ and $x \in G \setminus \{0\}$, and $p \in \mathbf{P}^* \setminus \mathbf{P}(x, G)$ then for some pure extension G' of G with $\text{rk}(G/G') = 1$ we have: $p \in \mathbf{P}^-(x, G')$.

For proving (*) for a given G, x let \hat{G} be the divisible hull of G and let

$$G_0 = \{y \in \hat{G} : \text{for some } n > 0, p^n y \in G\},$$

$G_1 = \{y \in \hat{G} : \text{for some } b \in \mathbb{Z}, b > 0 \text{ not divisible by } p \text{ we have } by \in G\}$. Clearly $G = G_0 \cap G_1$. We define the following subsets of $\hat{G} \times \mathbb{Q}$:

$$H_0 = \{(y, 0) : y \in G\} \text{ (so } G \text{ is isomorphic to } H_0)$$

$$H_1 = \{(p^n bx, p^n b) : b, n \in \mathbb{Z}\}$$

$$H_2 = \{(0, c_1/c_2) : c_1, c_2 \in \mathbb{Z} \text{ and } c_2 \text{ not divisible by } p\}.$$

Easily all three are additive subgroups of $\hat{G} \times \mathbb{Q}$ and $H_2 \cong \mathbb{Z}_{(p)}$. Let $G' = H_0 + H_1 + H_2$, a subgroup of $\hat{G} \times \mathbb{Q}$.

We claim that $G' \cap (\hat{G} \times \{0\}) = H_0$. The inclusion \supseteq should be clear. For the other direction let $z \in G' \cap (\hat{G} \times \{0\})$; as $z \in G'$ there are $(y, 0) \in H_0$, (so $y \in G$), $(p^n bx, p^n b) \in H_1$ (so $b \in \mathbb{Z}, n \in \mathbb{Z}$ and $x \in G$ is the constant from (*)) and $(0, c_1/c_2) \in H_2$ (so $c_1, c_2 \in \mathbb{Z}$ and p does not divide c_2) and integers a_0, a_1, a_2 such that $z = a_0(y, 0) + a_1(p^n bx, p^n b) + a_2(0, c_1/c_2)$ which means $z = (a_0 y + a_1 p^n bx, a_1 p^n b + a_2 c_1/c_2)$.

As $z \in \hat{G} \times \{0\}$ clearly $a_1 p^n b + a_2 c_1/c_2 = 0$, so as p does not divide c_2 , necessarily $a_1 p^n b$ is an integer, hence $a_1 p^n bx \in G$, hence $a_0 y + a_1 p^n bx \in G$ and hence $z \in G \times \{0\} = H_0$ as required.

It is easy to check now that H_0 is a pure subgroup of G' .

Also letting $y^* = (x, -1)$ clearly $(x, 0) - y^*$ is divisible by p^k for every $k < \omega$ (as

$(p^k x, p^k) \in H_1 \subseteq G'$ for every $k \in \mathbb{Z}$) and y^* is divisible by any integer b when b is not divisible by p (as $\frac{1}{b} y^* = (0, -1/b) \in H_2 \subseteq G'$).

Identifying $y \in G$ with $(y, 0) \in G$ we are done: G' is as required in $(*)$, with y^* witnessing “ $p \in \mathbf{P}^-(x, G')$ ”. $\square_{1.4}$

1.5 Claim. *If $G_1 \in \mathfrak{R}^{\text{rtf}}$ is full and $G_2 \in \mathfrak{R}^{\text{rtf}}$ and h is an embedding of G_1 into G_2 then:*

$$\text{for } x \in G_1 \setminus \{0\}, \mathbf{P}(x, G_1) = \mathbf{P}(h(x), G_2).$$

Proof. Without loss of generality h is the identity, now reflect using 1.4(1), 1.4(2) and the definition of full. $\square_{1.5}$

1.6 Conclusion. Assume

$$(*) \quad 2^{\aleph_0} < \mu^+ < \lambda = \text{cf}(\lambda) < \mu^{\aleph_0}.$$

Then there is no universal member in $\mathfrak{R}_\lambda^{\text{rtf}}$.

Proof. Let $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \aleph_0 \text{ and } \omega^2 \text{ divides } \delta\}$ be stationary and $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$ where each η_δ is an increasing ω -sequence of ordinals $< \delta$ with limit δ such that $\eta_\delta(n) - n$ is divisible by ω ; so $\delta_1 \neq \delta_2 \Rightarrow \text{Rang}(\eta_{\delta_1}) \cap \text{Rang}(\eta_{\delta_2})$ is finite. Let $\{p_n^* : n < \omega\}$ list the primes in the increasing order. Let $G_{\bar{\eta}}^0$ be the abelian group generated by $\{x_\alpha : \alpha < \lambda\} \cup \{y_\delta : \delta \in S\} \cup \{z_{\delta,n,\ell} : n, \ell < \omega\} \cup \{z_{\delta,n,m,\ell} : n, \ell < \omega, m \in \omega \setminus \{n\}\}$ freely except for the equations

$$p_n^* z_{\delta,n,\ell+1} = z_{\delta,n,\ell} \quad y_\delta - x_{\eta_\delta(n)} = z_{\delta,n,0}.$$

$$p_m^* z_{\delta,n,m,\ell+1} = z_{\delta,n,m,\ell+1}, x_{\eta_\delta} = z_{\delta,n,m,0}.$$

We can check that $G_{\bar{\eta}}^0 \in \mathfrak{R}_\lambda^{\text{rtf}}$.

Let $G_{\bar{\eta}} \in \mathfrak{R}_\lambda^{\text{rtf}}$ be a pure extension of $G_{\bar{\eta}}^0$ which is full (one exists by 1.4(3)). So

$$(*) \quad \text{if } h \text{ embeds } G_{\bar{\eta}} \text{ into } G \in \mathfrak{R}_\lambda^{\text{rtf}} \text{ then} \\ x \in G_{\bar{\eta}} \setminus \{0\} \Rightarrow \mathbf{P}(x, G_{\bar{\eta}}) = \mathbf{P}(h(x), G).$$

Hence the proof in [KjSh 455] works. $\square_{1.6}$

1.7 Remark. 1) Similarly the results on λ singular (i.e. Case 4) in [KjSh 455], hold for embedding (rather than pure embedding).

2) What about Case 5? If there is a family $\mathcal{P} \subseteq \{C \subseteq \mu^+ : \text{otp}(C) = \mu\}$ which guesses clubs (i.e. every club E of μ^+ contains one of them), the result holds.

3) On $\aleph_0 \leq \lambda < 2^{\aleph_0}$ see also in 3.16.

§2 THE EXISTENCE OF UNIVERSALS FOR
SEPARABLE REDUCED ABELIAN p -GROUPS

We here eliminate the very weak pcf assumption from the theorem of “no universal in $\mathfrak{K}_\lambda^{\text{rs}(\mathfrak{p})}$ ” when $\lambda > \beth_\omega$. Note that $\mathfrak{K}^{\text{rs}(\mathfrak{p})}$ is defined in 2.3(4).

In the first section we have eliminated the very weak pcf assumptions for the theorem concerning $\mathfrak{K}_\lambda^{\text{rtf}}$ (though the $\lambda = \text{cf}(\lambda) > \mu^+$ remains, i.e. we assume we are in Case 3). This was done using the “infinitely many primes”, so in the language of e.g. [KjSh 455] the invariant refers to one element x . This cannot be generalized to $\mathfrak{K}_\lambda^{\text{rs}(\mathfrak{p})}$. However, in [Sh 552, §5] we use an invariant on e.g. suitable groups and related stronger “combinatorial” ideals. We continue this, using combinatorial ideals closer to the algebraic ones to show that the algebraic is non-trivial.

We rely on the “GCH” proved in [Sh 460] hence the condition “ $\lambda > \beth_\omega$ ” is used.

2.1 Definition. 1) For $\bar{\lambda} = \langle \lambda_\ell : \ell < \omega \rangle$ and $\bar{t} = \langle t_\ell : \ell < \omega \rangle$ (with $1 < t_\ell < \omega$) we define $J_{\bar{t}, \bar{\lambda}}^4$.

It is a family of subsets A of $\prod_{\ell < \omega} [\lambda_\ell]^{t_\ell}$ such that:

- (*) for every large enough $\ell < \omega$, for every $B \in [\lambda_\ell]^{\aleph_0}$ for some $k \in (\ell, \omega)$ we cannot find

$$\langle \nu_\eta : \eta \in \prod_{i \in [\ell, k)} [\omega]^{t_i} \rangle$$

such that

(a) $\nu_\eta \in A$

(b) if $\eta_1, \eta_2 \in \prod_{i \in [\ell, k)} [\omega]^{t_i}$, $\ell \leq m \leq k$ and $\eta_1 \upharpoonright [\ell, m) = \eta_2 \upharpoonright [\ell, m)$ then

$$\nu_{\eta_1} \upharpoonright m = \nu_{\eta_2} \upharpoonright m; \text{ hence}$$

$$\nu_{\eta_1} \upharpoonright \ell = \nu_{\eta_2} \upharpoonright \ell \text{ for } \eta_1, \eta_2 \in \prod_{i \in [\ell, k)} [\omega]^{t_i}$$

(c) if $\eta_0 \in \prod_{i \in [\ell, k)} [\omega]^{t_i}$ and $\ell \leq m < k$ then for some $E \in [\lambda_m]^{\aleph_0}$ we have

$$[E]^{t_m} = \{ \nu_\eta(m) : \eta \in \prod_{i \in [\ell, k)} [\omega]^{t_i} \text{ and } \eta \upharpoonright m = \eta_0 \upharpoonright m \}$$

$$\text{and } m = \ell \Rightarrow E = B.$$

2) Let

$$J_{\bar{t}, \bar{\lambda}, < \theta}^4 = \left\{ A \subseteq \prod_{\ell < \omega} [\lambda_\ell]^{t_\ell} : \text{for some } \alpha < \theta \text{ and } A_\beta \in J_{\bar{t}, \bar{\lambda}}^4 \text{ for } \beta < \alpha \right.$$

$$\left. \text{we have } A \subseteq \bigcup_{\beta < \alpha} A_\beta \right\}.$$

When $\theta = \kappa^+$, we may write κ instead of $< \theta$.

2.2 *Fact.* 1) $J_{\bar{t}, \bar{\lambda}, \theta}^4$ is a θ -complete ideal.

2) If $\ell < \omega \Rightarrow \lambda_\ell > \beth_{t_\ell-1}(\theta)$ then the ideal $J_{\bar{t}, \bar{\lambda}, \theta}^4$ is proper (where $\beth_0(\theta) = \theta, \beth_{n+1}(\theta) = 2^{\beth_n(\theta)}$, and for α limit $\beth_\alpha(\theta) = \theta + \sum_{\beta < \alpha} 2^{\beth_\beta(\theta)}$).

Proof. 1) Trivial.

2) Let for $\ell < \omega$

$$ERI_{\lambda_\ell}^{n_\ell} = \{A \subseteq [\lambda_\ell]^{t_\ell} : \text{for some } F : [\lambda_\ell]^{t_\ell} \rightarrow \theta \text{ there is no } B \in [\lambda_\ell]^{\aleph_0} \text{ such that } F \upharpoonright [B]^{t_\ell} \text{ is constant and } [B]^{t_\ell} \subseteq A\}.$$

So this is a θ^+ -complete ideal. It is non-trivial by Erdős-Rado theorem (we use it similarly in [Sh 620, §1]). Now we shall prove that the ideal $J_{\bar{t}, \bar{\lambda}, \theta}^4$ is proper. So assume $\prod_{\ell < \omega} [\lambda_\ell]^{t_\ell} = \bigcup_{i < \theta} X_i$ and $X_i \in J_{\bar{t}, \bar{\lambda}}^4$ for each $i < \theta$ and we shall get a contradiction. Let

$$X_i^+ = \{\eta \in \prod_{\ell < \omega} [\lambda_\ell]^{t_\ell} : \text{for every } \ell < \omega \text{ for some } \eta' \in X_i \text{ we have } \eta \upharpoonright \ell = \eta' \upharpoonright \ell\}.$$

(i.e. the closure of X_i). So $X_i^+ \subseteq \prod_{\ell < \omega} [\lambda_\ell]^{t_\ell} = \prod_{\ell < \omega} \text{Dom}(ERI_{\lambda_\ell}^{t_\ell})$ is closed, and those ideals are θ^+ -complete and $\prod_{\ell < \omega} \text{Dom}(ERI_{\lambda_i}^{t_\ell}) = \bigcup_{i < \theta} X_i^+$. Hence (Rubin-Shelah [RuSh 117], [Sh:f, Ch.XI, 3.5(2)] with $H_\alpha = X_i^+$) we can find T such that:

- (a) $T \subseteq \bigcup_{m < \omega} \prod_{\ell < m} [\lambda_\ell]^{t_\ell}$
- (b) T is closed under initial segments
- (c) $< > \in T$
- (d) if $\nu \in T$ and $\ell g(\nu) = \ell$ then $\{u \in [\lambda_\ell]^{t_\ell} : \nu \hat{\ } \langle u \rangle \in T\} \in (ERI_{\lambda_\ell}^{t_\ell})^+$
- (e) for some $i < \theta$, $\lim(T) \subseteq X_i^+$.

(Here, $\lim(T) = \{\nu \in \prod_{\ell < \omega} [\lambda_\ell]^{t_\ell} : (\forall m < \omega) \nu \upharpoonright m \in T\}$).

By the definition of the ideal we get more than required (for every k in place of “some k ” in (*) of Definition 2.1). $\square_{2.2}$

Remark. So we could have used the stronger ideal defined implicitly in 2.2, i.e. $J_{\bar{t}, \bar{\lambda}, \theta}^5 = \{X \subseteq \prod_{\ell < \omega} \lambda_\ell : \text{we can find } \alpha < \theta \text{ and } X_i \subseteq X \text{ for } i < \alpha \text{ such that } X =$

$\bigcup_{i < \alpha} X_i$ and for each i and T satisfying clauses (a) – (d) from the proof of 2.2 there is $T' \subseteq T$ satisfying clauses (a) – (d) such that $\lim(T)$ is disjoint to the closure of X_i . Of course, we can also replace $\text{ERI}_{\lambda_\ell}^{t_\ell}$ by various variants.

We recall from [Sh 552, 5.1]

2.3 Definition. ([Sh 552, 5.1]) 1) For $\bar{\mu} = \langle \mu_n : n < \omega \rangle$ let $B_{\bar{\mu}}$ be the following direct sum of cyclic p -groups. Let K_α^n be a cyclic group of order p^{n+1} generated by x_α^n and let $B_{\mu_n} = \bigoplus_{\alpha < \mu_n} K_\alpha^n$ and $B_{\bar{\mu}} = \bigoplus_{n < \omega} B_{\mu_n}$, i.e. $B_{\bar{\mu}}$ is the abelian group generated by $\{x_\alpha^n : n < \omega, \alpha < \mu_n\}$ freely except that $p^{n+1} x_\alpha^n = 0$.

Moreover, let $B_{\bar{\mu} \upharpoonright n} = \bigoplus \{K_\alpha^m : \alpha < \mu_n, m < n\} \subseteq B_{\bar{\mu}}$
(these groups are in $\mathfrak{K}_{\leq \sum_n \mu_n}^{\text{rs}(p)}$).

Let $\hat{B}_{\bar{\mu}}$ be the p -torsion completion of $B_{\bar{\mu}}$ (i.e. completion under the norm $\|x\| = \min\{2^{-n} : p^n \text{ divides } x\}$ but putting only the torsion elements, see Fuchs [Fu]. Note that $\hat{B}_{\bar{\mu}}$ is the torsion part of the p -adic completion of $B_{\bar{\mu}}$).

2) Let $I_{\bar{\mu}, < \theta}^1 = I_{\bar{\mu}, \theta}^1[p]$ be the ideal on $\hat{B}_{\bar{\mu}}$ (depending on the choice of $\langle K_\alpha^n : \alpha < \mu_n; n < \omega \rangle$ or actually $\langle B_{\bar{\mu} \upharpoonright n} : n < \omega \rangle$) consisting of unions of $< \theta$ members of $I_{\bar{\mu}}^0$, where

$$I_{\bar{\mu}}^0 = I_{\bar{\mu}, \theta}^0[p] = \{A \subseteq \hat{B}_{\bar{\mu}} : \text{for every large enough } n, \text{ we have } c\ell_{\hat{B}_{\bar{\mu}}}(\langle A \rangle_{\hat{B}_{\bar{\mu}}}) \cap B_{\bar{\mu}} \subseteq B_{\bar{\mu} \upharpoonright n}\}$$

($c\ell_{\hat{B}_{\bar{\mu}}}$ is defined in 3) below).

When $\theta = \kappa^+$ instead of $< \theta$ we may write κ . If $\mu_n = \mu$, we may write μ instead of $\bar{\mu}$.

3) For $X \subseteq \hat{B}_{\bar{\mu}}$, recall

$$c\ell_{\hat{B}_{\bar{\mu}}}(X) = \{x : (\forall n)(\exists y \in X)(x - y \in p^n \hat{B}_{\bar{\mu}})\}.$$

4) Let $\mathfrak{K}^{\text{rs}(p)}$ be the family of pure subgroups of some $\hat{B}_{\bar{\mu}}$.

5) If p is not clear from the context, we may write $B_{\bar{\mu}}^p, \hat{B}_{\bar{\mu}}^p$, etc.

2.4 Claim. Assume $\bar{\mu} = \langle \mu_n : n < \omega \rangle, \bar{t} = \langle t_\ell : \ell < \omega \rangle, t_\ell = p$ and the ideal $J_{\bar{t}, \bar{\mu}, \theta}^4$ is proper (so $\mu_n \geq \beth_{p-1}(\theta)^+$ is enough by 2.2(2)). Then the ideal $I_{\bar{\mu}, \theta}^1$ is proper (and $I_{\bar{\mu}, \theta}^1$ is a θ^+ -complete ideal).

Proof. We define a function h from $\prod_{\ell < \omega} [\lambda_\ell]^{t_\ell}$ into $\hat{B}_{\bar{\mu}}$. We let

$$h(\eta) = \Sigma \{p^n x_\beta^n : \beta \in \eta(n) \text{ and } n < \omega\} \in \hat{B}_{\bar{\mu}}[p].$$

Clearly h is one to one and it suffices to prove

(*) if $X \in (J_{\bar{t}, \bar{\mu}, \theta}^4)^+$ then $h''(X)$ belongs to $(I_{\bar{\mu}, \theta}^1)^+$.

So assume $X \in (J_{\bar{t}, \bar{\lambda}, \theta}^4)^+$ is given and suppose toward contradiction that $h''(X) \in I_{\bar{\mu}, \theta}^1$. So we can find $\langle Y_i : i < \theta \rangle$ such that for such $i < \theta$ we have $Y_i \in I_{\bar{\mu}}^0$ and $h(X) \subseteq \bigcup_{i < \theta} Y_i$. Let $X_i = h^{-1}(Y_i)$. So $h(X_i) \subseteq Y_i \in I_{\bar{\mu}}^0$ and hence $h(X_i) \in I_{\bar{\mu}}^0$, but as $J_{\bar{t}, \bar{\lambda}, \theta}^4$ is θ^+ -complete and $X \in (J_{\bar{t}, \bar{\lambda}, \theta}^4)^+$ necessarily for some $i < \theta$, $X_i \in (J_{\bar{t}, \bar{\lambda}, \theta}^4)^+$, so without loss of generality $h''(X) \in I_{\bar{\mu}}^0$. By the definition of $I_{\bar{\mu}}^0$, for some $n(*) < \omega$ we have

$$(*) \quad B_{\bar{\mu}} \cap c\ell(h''(X)) \subseteq B_{\bar{\mu} \upharpoonright n(*)}.$$

On the other hand, as $X \in (J_{\bar{t}, \bar{\mu}, \theta}^4)^+$, it is $\notin J_{\bar{t}, \bar{\mu}}^4$ so from definition 2.1(1) of $J_{\bar{t}, \bar{\mu}}^4$ we can find $\langle B_n : n \in \Gamma \rangle$ such that:

- (a) $\Gamma \in [\omega]^{\aleph_0}$ and $B_n \in [\lambda_n]^{\aleph_0}$
- (b) for $n \in \Gamma$, for every $k \in (n, \omega)$ we can find $\langle \nu_{\eta}^{n,k} : \eta \in \prod_{\ell \in [n, k)} [\omega]^{t_\ell} \rangle$ as in (a)-(c) of Definition 2.1(1) with n, B_n, k here standing for ℓ, B, k there.

For $m \in (n, k]$ and $\eta \in \prod_{\ell \in [n, m)} [\omega]^{t_\ell}$ we let $\nu_{\eta}^{n,k}$ be $\nu_{\eta_1}^{n,k} \upharpoonright m$ whenever

$\eta \triangleleft \eta_1 \in \prod_{\ell \in [n, k)} [\omega]^{t_\ell}$ (by clause (b) in (*) of 2.1 it is well defined). Fix $n \in \Gamma$ and

$k \in [n, \omega)$ for awhile. Let $A_\eta = A_\eta^{n,k} \in [\lambda_m]^{\aleph_0}$ be such that $\{\nu_{\eta \restriction \langle u \rangle}^{n,k}(m) : u \in [\omega]^{t_m}\} = [A_\eta]^{t_m}$ and without loss of generality (otp stands for “the order type”)

$$(*) \quad \text{otp}(A_\eta) = \omega \text{ and } \nu_{\eta \restriction \langle u \rangle}^{n,k}(m) = \text{OP}_{A_\eta, \omega}(u)$$

(where $\text{OP}_{A_\eta, \omega}(i) = \alpha$ iff $i = \text{otp}(A_\eta \cap \alpha)$).

Now for $m \in (n, k]$ and $\eta \in \prod_{\ell \in [n, m)} [w]^{t_\ell}$ we define

$$y_\eta = y_\eta^{n,k} = \sum \{h(\nu_\rho^{n,k}) : \eta \trianglelefteq \rho \in \prod_{\ell \in [n, k)} [\omega]^{t_\ell} \text{ and } (\forall \ell)[\ell g(\eta) \leq \ell < k \rightarrow \rho(\ell) \subseteq [0, t_\ell]]\}$$

where \trianglelefteq denotes being an initial segment. So $y_\eta \in \hat{B}_{\bar{\mu}}$ and we shall prove by downward induction on $m \in (n, k]$ that for every $\eta \in \prod_{\ell \in [n, m)} [w]^{t_\ell}$ we have

$$\boxtimes \quad y_\eta = \left(\prod_{\ell=m}^{k-1} (t_\ell + 1) \right) \times \left(\sum_{\ell < m} \sum_{\alpha \in \nu_\eta^{n,k}(\ell)} p^\ell x_\alpha^\ell \right) \bmod p^k \hat{B}_{\bar{\mu}}.$$

Case 1: $m = k$.

In this case the product $\prod_{\ell=m}^{k-1} (t_\ell + 1)$ is just 1, so the equation says

$$y_\eta = \sum_{\ell < m} \sum_{\alpha \in \nu_\eta^{n,k}(\ell)} p^\ell x_\alpha^\ell \bmod p^k \hat{B}_{\bar{\mu}}.$$

Now the expression for y_η is

$$\begin{aligned}
& \sum \{h(\nu_\rho^{n,k}) : \eta \trianglelefteq \rho \in \prod_{\ell \in [n,k]} [\omega]^{t_\ell} \text{ and } (\forall \ell)[m \leq \ell < k \Rightarrow \rho(\ell) \subseteq [0, t_\ell]]\} \\
&= h(\nu_\eta^{n,k}) = \sum_{\ell < \omega} \sum_{\alpha \in \nu_\eta^{n,k}(\ell)} p^\ell x_\alpha^\ell \\
&= \sum_{\ell < m} \sum_{\alpha \in \nu_\eta^{n,k}(\ell)} p^\ell x_\alpha^\ell + p^k \left(\sum_{\ell \in [k, \omega)} \sum_{\alpha \in \nu_\eta^{n,k}(\ell)} p^{\ell-k} x_\alpha^\ell \right)
\end{aligned}$$

so the equality is trivial.

Case 2: $n < m < k$.

Here (with equalities in the equation being in $\hat{B}_{\bar{\mu}}$, modulo $p^k \hat{B}_{\bar{\mu}}$), we have:

$$\begin{aligned}
y_\eta &= \\
& \quad \text{[by the definition of } y_\eta, y_{\eta^\wedge \langle u \rangle}] \\
&= \sum \{y_{\eta^\wedge \langle u \rangle} : u \in [\{0, \dots, t_m\}]^{t_m}\} = \\
& \quad \text{[by the induction hypothesis]} \\
&= \sum \left\{ \left(\prod_{\ell=m+1}^{k-1} (t_\ell + 1) \right) \left(\sum_{\ell < m+1} \sum_{\alpha \in \nu_{\eta^\wedge \langle u \rangle}^{n,k}(\ell)} p^\ell x_\alpha^\ell \right) : u \in [\{0, \dots, t_m\}]^{t_m} \right\} \\
& \quad \text{[by dividing the sum } \sum_{\ell < m+1} \text{ into } \sum_{\ell < m} \text{ and } \sum_{\ell=m} \text{ and noting what } \nu_{\eta^\wedge \langle u \rangle}^{n,k}(m) \text{ is]} \\
&= \sum \left\{ \left(\prod_{\ell=m+1}^{k-1} (t_\ell + 1) \right) \left(\sum_{\ell < m} \sum_{\alpha \in \nu_{\eta^\wedge \langle u \rangle}^{n,k}(\ell)} p^\ell x_\alpha^\ell \right) : u \in [\{0, \dots, t_m\}]^{t_m} \right\} \\
& \quad + \sum \left\{ \left(\prod_{\ell=m+1}^{k-1} (t_\ell + 1) \right) \sum_{\alpha \in OP_{\omega, A_\eta}(u)} p^m x_\alpha^m : u \in [\{0, \dots, t_m\}]^{t_m} \right\} = \\
& \quad \text{[in the second sum, we collect together the terms with } x_\alpha^m] \\
&= \sum \left\{ \left(\prod_{\ell=m+1}^{k-1} (t_\ell + 1) \right) \left(\sum_{\ell < m} \sum_{\alpha \in \nu_\eta^{n,k}(\ell)} p^\ell x_\alpha^\ell \right) : u \in [\{0, \dots, t_m\}]^{t_m} \right\} \\
& \quad + \sum \left\{ \left(\prod_{\ell=m+1}^{k-1} (t_\ell + 1) \right) (p^m x_\alpha^m) | \{u : u \in [\{0, \dots, t_m\}]^{t_m} \text{ and } |\alpha \cap A_\eta| \in u\} : \right. \\
& \quad \left. \alpha \text{ is a member of } A_\eta, \text{ moreover } |\alpha \cap A_\eta| \leq t_m \right\}
\end{aligned}$$

$$\begin{aligned}
&= \left(\prod_{\ell=m+1}^{k-1} (t_\ell + 1) \right) \left(\sum_{\ell < m} \sum_{\alpha \in \nu_\eta^{n,k}(\ell)} p^\ell x_\alpha^\ell \right) \times |\{u : u \in [\{0, \dots, t_m\}]^{t_m}\}| \\
&+ \sum \left\{ \left(\prod_{\ell=m+1}^{k-1} (t_\ell + 1) \right) (p^m x_\alpha^m) \cdot ((t_m + 1) - 1) : \alpha \in A_\eta, |\alpha \cap A_\eta| \leq t_m \right\} = \\
&\quad [\text{remember } t_m = p \text{ and } p^{m+1} x_\alpha^m = 0] \\
&= (t_m + 1) \left(\prod_{\ell=m+1}^{k-1} (t_\ell + 1) \right) \sum_{\ell < m} \sum_{\alpha \in \nu_\eta^{n,k}(\ell)} p^\ell x_\alpha^\ell + 0 \\
&= \left(\prod_{\ell=m}^k (t_\ell + 1) \right) \left(\sum_{\ell < m} \sum_{\alpha \in \nu_\eta^{n,k}(\ell)} p^\ell x_\alpha^\ell \right).
\end{aligned}$$

Hence we have finished the proof of \boxtimes .

So, as for $n \in \Gamma$, B_n serves for every $k \in (n, \omega)$, if $u_1, u_2 \in [B_n]^{t_n}$ are distinct then $y_{\langle u_1 \rangle} - y_{\langle u_2 \rangle}$ contradicts $(*)$. $\square_{2.4}$

Recall

2.5 Definition. 1) Let I be an ideal on κ (or just $I \subseteq \mathcal{P}(\kappa)$ closed downward, $I^+ = \mathcal{P}(\kappa) \setminus I$), then we let:

$$\begin{aligned}
\mathbf{U}_I(\lambda) = & \text{Min}\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^{\leq \kappa} \text{ and for every } f \in {}^\kappa \lambda \\
& \text{for some } a \in \mathcal{P} \text{ we have } \{i < \kappa : f(i) \in a\} \in I^+\}.
\end{aligned}$$

2) For $\sigma \leq \theta \leq \mu \leq \lambda$ let $\text{cov}(\lambda, \mu, \theta, \sigma) = \text{Min}\{\lambda + |\mathcal{P}| : \mathcal{P} \text{ is a family of subsets of } \lambda \text{ each of cardinality } \mu \text{ such that any } X \subseteq \lambda \text{ of cardinality } < \theta \text{ is included in the union of } < \sigma \text{ members of } \mathcal{P}\}.$

2.6 Claim. 1) For every $\lambda \geq \beth_\omega$, for some $\theta < \beth_\omega$ for every $\mu \in (\beth_{p-1}(\theta), \beth_\omega)$ we have (letting $\mu_n = \mu$) $\mathbf{U}_{I_{\bar{\mu}, \theta}^1}(\lambda) = \lambda$ (hence $\mathbf{U}_{I_{\bar{\mu}}^0}(\lambda) = \lambda$).

2) If $\text{cf}(\lambda) > \beth_\omega$, then for some $\theta < \beth_\omega$, for every $\mu \in (\beth_{p-1}(\theta), \beth_\omega)$ and $\lambda' < \lambda$ we have $\mathbf{U}_{I_{\bar{\mu}, \theta}^1}(\lambda') < \lambda$.

Proof. By 2.4, $I_{\bar{\mu}, \theta}$ is a θ -complete proper ideal on a set of cardinality μ^{\aleph_0} , for any μ, θ as in the assumptions. By [Sh 460] for each $\lambda' \leq \lambda$ for some $\theta = \theta[\lambda'] < \beth_\omega$ for every $\mu \in (\theta, \beth_\omega)$ we have $\text{cov}(\lambda', \mu^+, \mu^+, \theta) = \lambda'$, i.e. there is $\mathcal{P}_\mu \subseteq [\lambda']^\mu$ of cardinality $\leq \lambda'$ such that: if $Y \in [\lambda']^{\leq \mu}$ then Y is included in the union of $< \theta$ members of \mathcal{P}_μ . As $I_{\bar{\mu}, \theta}^1$ is a θ^+ -complete ideal on a set of cardinality μ it follows that $\mathbf{U}_{I_{\bar{\mu}, \theta}^1}(\lambda') \leq \lambda' \times |\mathcal{P}_\mu| = \lambda'$ (and trivially $\mathbf{U}_{I_{\bar{\mu}, \theta}^1}(\lambda) \geq \lambda$). For some $\theta < \beth_\omega$, for arbitrarily large $\lambda' < \lambda$, $\theta[\lambda'] \leq \theta$ and $\theta[\lambda] \leq \theta$; clearly we are done. $\square_{2.6}$

2.7 Conclusion. If $\beth_\omega \leq \mu^+ < \lambda = \text{cf}(\lambda) < \mu^{\aleph_0}$, then in $\mathfrak{K}_\lambda^{\text{rs}(\mathfrak{p})}$ there is no universal member.

Proof. By 2.6 and [Sh 552, 5.9].

Moreover

2.8 Claim. *Assume*

- (a) $\prod_{\ell < \omega} \kappa_\ell < \mu < \lambda = \text{cf}(\lambda) \leq \lambda' < \mu^{\aleph_0}$
- (b) $\mu^+ < \lambda$ or at least for some \mathcal{P}
 - (*) $\mathcal{P} \quad |\mathcal{P}| = \lambda \quad \& \quad (\forall a \in \mathcal{P})(a \subseteq \lambda \quad \& \quad \text{otp}(a) = \mu)$
 $\quad \& \quad (\forall E)(E \text{ a club of } \lambda \rightarrow (\exists a \in \mathcal{P})(a \subseteq E))$
- (c) $\lambda' = \mathbf{U}_{I_{\bar{\kappa}}^0}(\lambda) < \mu^{\aleph_0}$ where $\bar{\kappa} = \langle \kappa_\ell : \ell < \omega \rangle$ and note that $I_{\bar{\kappa}}^0$ depends on the prime p .

Then we can find reduced separable abelian p -groups, $G_\alpha \in \mathfrak{K}_\lambda^{\text{rs}(\mathfrak{p})}$ for $\alpha < \mu^{\aleph_0}$ such that for every reduced separable abelian p -group G of cardinality λ' we have:

some G_α is not embeddable into G ; also the number of ordinals $\alpha < \mu^{\aleph_0}$ such that ?

Moreover, each G_α is slender, i.e. there is no homomorphism from \mathbb{Z}^ω into G_α with range of infinite rank.

Proof. Same proof as that of [Sh 552, 5.9], [Sh 552, 7.5].

§3 NON-EXISTENCE OF UNIVERSALS FOR \aleph_1 -FREE ABELIAN GROUPS

The first section dealt with $\mathfrak{K}_\lambda^{\text{rtf}}$ improving [Sh 552]. But the groups used there are “almost divisible”. So what occurs if we replace $\mathfrak{K}^{\text{rtf}}$ by a variant avoiding this? We suggest to consider the \aleph_1 -free abelian groups where type arguments like in §1 break down. So the proof of [Sh 552] becomes relevant and it is natural to improve it as in §2 (which deals with $\mathfrak{K}^{\text{rs}(p)}$), for diversity we use a stronger ideal. We have not looked at the problem for \aleph_1 -free abelian groups of cardinality λ when $\aleph_0 < \lambda < 2^{\aleph_0}$.

So we concentrate here on torsion free (abelian) groups.

3.1 Definition. 1) Let $\bar{t} = \langle t_\ell : \ell < \omega \rangle, 2 \leq t_\ell < \omega$. For abelian group H , the \bar{t} -valuation is

$$\|x\|_{\bar{t}} = \inf\{2^{-m} : \prod_{\ell < m} t_\ell \text{ divide } x \text{ (in } G)\}.$$

This is a semi-norm. Remember $d_{\bar{t}}(x, y) = \|x - y\|_{\bar{t}}$. This semi-norm induces a topology which we call the \bar{t} -adic topology.

If $t_\ell = p$ for $\ell < \omega$ we may write p instead of \bar{t} .

2) Let $\text{cl}_{\bar{t}}(A, H)$ be the closure of A in H under the \bar{t} -adic topology.

Let $PC_H(X)$ be the pure closure of X in H . Moreover $PC_H^p(X)$ is the p -adic closure in H of the subgroup of H which X generates.

3) Let $\mathfrak{K}^{\text{rtf}}[\bar{t}]$ be the class of \bar{t} -reduced torsion free abelian groups, i.e. the $G \in \mathfrak{K}^{\text{rtf}}$ such that $\bigcap_{n < \omega} \left(\prod_{i < n} t_i \right) G = \{0\}$ hence $\| - \|_{\bar{t}}$ induces a Hausdorff topology.

(Inversely if G is torsion free with the \bar{t} -adic topology Hausdorff then $G \in \mathfrak{K}^{\text{rtf}}[\bar{t}]$.)

4) If the \bar{t} -adic topology is Hausdorff, then $G^{[\bar{t}]}$ is the completion of G by $\| - \|_{\bar{t}}$.

If $t_\ell = 2 + \ell$, this is the \mathbb{Z} -adic completion.

The following continues the analysis in [Sh 552, 1.1] (which deals with $\mathfrak{K}^{\text{rs}(p)}$) [Sh 552, 1.5] (which deals with $\mathfrak{K}^{\text{rtf}}$).

3.2 Definition. We say G has \bar{t} -density μ if it has a pure subgroup of cardinality $\leq \mu$ which is \bar{t} -dense, i.e. dense in the \bar{t} -adic topology, but has no such subgroup of cardinality $< \mu$.

3.3 Proposition. Suppose that

- (α) $\mu \leq \lambda \leq \mu^{\aleph_0}$
- (β) G is an \aleph_1 -free abelian group, $|G| = \lambda$
- (γ) \bar{t} is as in 3.1 such that $(\forall \ell)(\exists m > \ell) (t_\ell \text{ divides } t_m)$.

Then there is an \aleph_1 -free group H such that $G \subseteq H, |H| = \lambda$ and H has \bar{t} -density μ .

Proof. Choose $\lambda_n < \mu$ for $n < \omega$ such that $\prod_{n < \omega} \lambda_n \geq \lambda, \mu \geq \sum_{n < \omega} \lambda_n, 2\lambda_n < \lambda_{n+1}$ (so $\lambda_n > 0$ may be finite). Let $\{x_i : i < \lambda\}$ list the elements of G . Let $\lambda'_{n+1} =$

$\lambda_{n+1}, \lambda'_0 = \mu$ if $\mu > \sum_{n < \omega} \lambda_n$ and $\lambda'_n = \lambda_n$ for all n if $\mu = \sum_{n < \omega} \lambda_n$. Let $\eta_i \in \prod_{n < \omega} \lambda'_n$ for $i < \lambda$ be distinct such that $\eta_i(n+1) \geq \lambda_n$ and $i \neq j \Rightarrow (\exists m)(\forall n)[m \leq n \Rightarrow \eta_i(n) \neq \eta_j(n)]$, and $\{\eta_i(0) : i < \lambda\} = \lambda'_0$. Without loss of generality $\mu = \{\eta_i(n) : i < \lambda, n < \omega\}$. Let H be generated by G, x_i^m (for $i < \lambda'_m, m < \omega$), y_i^n (for $i < \lambda, n < \omega$) freely except for

- (a) the equations of G
- (b) $y_i^0 = x_i$ ($\in G$)
- (c) $t_n y_i^{n+1} + y_i^n = x_{\eta_i(n)}^n$.

Fact A: H extends G and is torsion free.

Proof. H can be embedded into the divisible hull of $G \times F$, where F is the abelian group generated freely by $\{x_\alpha^m : m < \omega \text{ and } \alpha < \lambda'_m\}$.

Fact B: H is \aleph_1 -free and moreover H/G is \aleph_1 -free.

Proof. Let K be a countable pure subgroup of H . Now without loss of generality K is generated by

- (i) $K_1 = \{x_i : i \in I\}$ is a pure subgroup of G , where I is some countably infinite subset of λ , and so $G \supseteq K_1$,
- (ii) y_i^m, x_j^n for $i \in I, m < \omega$ and $(n, j) \in J$, where $J \subseteq \omega \times \lambda$ is countable and

$$i \in I, n < \omega \Rightarrow (n, \eta_i(n)) \in J.$$

Moreover, the equations holding among those elements are deducible from the equations of the form

- (a)⁻ equations of K_1
- (b)⁻ $y_i^0 = x_i$ for $i \in I$
- (c)⁻ $t_n y_i^{n+1} + y_i^n = x_{\eta_i(n)}^n$ for $i \in I, n < \omega$.

We can find $\langle k_i : i < \omega \rangle$ such that $[i \neq j \ \& \ i \in I \ \& \ j \in I \ \& \ n \geq k_i \ \& \ n \geq k_j \ \& \ i \neq j \Rightarrow \eta_i(n) \neq \eta_j(n)]$.

Now we know that K_1 is free (being a countable subgroup of G), and it suffices to prove that K/K_1 is free. But K/K_1 is freely generated by $\{y_i^n : i \in I \text{ and } n > k_i\} \cup \{x_\alpha^n : (n, \alpha) \in J \text{ but for no } i \in I \text{ do we have } n > k_i, \eta_i(n) = \alpha\}$. So K is free.

Fact C: $H_0 = \langle x_i^n : n < \omega, i < \lambda'_n \rangle_H$ satisfies:

- (a) $i < \lambda \Rightarrow d_{\bar{t}}(x_i, H_0) = \inf\{d_{\bar{t}}(x_i, z) : z \in H_0\} = 0$
- (b) $x \in G \Rightarrow d_{\bar{t}}(x, H_0) = 0$
- (c) $x \in H \Rightarrow d_{\bar{t}}(x, H_0) = 0$.

Proof. First note that

$$(*)_1 \quad Y = \{x \in H : d_{\bar{t}}(x, H_0) = 0\}$$

is a subgroup of H . Also for every $i < \lambda$ and n

$$\begin{aligned} (*)_2 \quad y_i^n &= x_{\eta_i(n)}^n + t_n y_i^{n+1} = x_{\eta_i(n)}^n + t_n x_{\eta_i(n+1)}^{n+1} + t_n t_{n+1} y_i^{n+2} \\ &= \sum_{k=n}^m \left(\prod_{\ell=n}^{k-1} t_\ell \right) x_{\eta_i(k)}^k + \left(\prod_{\ell=n}^k t_\ell \right) y_i^{k+1} \end{aligned}$$

(prove by induction on $m \geq n$), and note that as $(\forall \ell)(\exists m > \ell)(t_i \text{ divides } t_m)$ necessarily $(\forall \ell)(\exists^\infty m)(t_\ell \text{ divides } t_m)$ hence $(\forall k)(\exists^\infty m)(\prod_{i \leq \ell} t_\ell \text{ divides } \prod_{i=k}^m t_i)$. Now

$(*)_2$ implies

$$(*)_3 \quad y_i^n \in Y.$$

But $x_i = y_i^0$ and hence clause (a) holds, so as $\{x_i : i < \lambda\}$ is dense in G also clause (b) holds. So $G \subseteq Y$ (by clause (b)), and $x_\alpha^n \in Y$ (as $H_0 \subseteq Y$ and the choice of H_0) and $y_i^n \in Y$ (by $(*)_3$).

By $(*)_1$ clearly $Y = H$, as required in clause (c).

Fact D: $|H| = \lambda$.

Fact E: The \bar{t} -density of H is μ .

Proof. It is $\leq \mu$ as H_0 has cardinality μ and is \bar{t} -dense in H , it is $\geq \mu$, as we now show. Otherwise let $X^* \subseteq H$ be \bar{t} -dense with $|X^*| < \mu$, so for some $n, |X^*| < \lambda'_n$.

Define a function h with domain the generators of H listed above, into H . Let $h(x) = 0$ if $x \in G$; $h(x_\alpha^m) = 0$ if $m \neq n, \alpha < \lambda_m$; $h(x_\alpha^m) = x_\alpha^m$ if $m = n, \alpha < \lambda_m$; $h(y_i^m) = 0$ if $m > n$; $h(y_i^m) = -x_{\eta_i(n)}^n$ if $m = n$; $h(y_i^m) = \prod_{\ell=m}^{n-1} t_\ell \times h(y_i^n)$ if $m < n$.

This function preserves the equations defining H and hence induces a homomorphism \hat{h} from H onto $\langle \text{Rang}(h) \rangle_H = \langle \{x_\alpha^n : \alpha < \lambda'_n\} \rangle_H$. Clearly $h(h(x)) = h(x)$ for the generators hence $\hat{h} \circ \hat{h} = \hat{h}$. Hence $\langle \{x_\alpha^n : \alpha < \lambda'_n\} \rangle_H$ is a direct summand of H and hence the $d_{\bar{t}}$ -density of H is at least the $d_{\bar{t}}$ -density of $\langle \{x_\alpha^n : \alpha < \lambda'_n\} \rangle_H$ which is λ'_n . $\square_{3.3}$

We define variants of Definition 2.1.

3.4 Definition. For $\bar{\lambda} = \langle \lambda_\ell : \ell < \omega \rangle, \bar{t} = \langle t_\ell : \ell < \omega \rangle, 2 \leq t_\ell < \omega$, we let

$$J_{\bar{t}, \bar{\lambda}}^5 = \left\{ X \subseteq \prod_{\ell < \omega} [\lambda_\ell]^{t_\ell} : \text{we cannot find } m(*) < \omega, \bar{Y} = \langle Y_m : m < \omega \text{ and } m \geq m(*) \rangle, \right.$$

$\bar{A}^m = \langle A_\eta : \eta \in Y_m \rangle$ such that:

- (a) $Y_m \subseteq \prod_{\ell < m} [\lambda_\ell]^{t_\ell}$
- (b) $Y_{m(*)} \subseteq \prod_{\ell < m(*)} [\lambda_\ell]^{t_\ell}$ is a singleton
- (c) $\langle A_\eta : \eta \in Y_m \rangle$ is a sequence of pairwise disjoint subsets of λ_m of order type ω
- (d) $Y_{m+1} = \{\eta^\wedge \langle u \rangle : \eta \in Y_m \text{ and } u \in [A_\eta]^{t_m}\}$
- (e) $Y_m \subseteq \{\nu \upharpoonright m : \nu \in X\}$

$J_{\bar{t}, \bar{\lambda}}^6$ is defined similarly but $m(*) = 0$,

$$J_{\bar{t}, \bar{\lambda}, < \theta}^\ell = \left\{ X : \text{for some } \alpha < \theta \text{ and } X_\beta \in J_{\bar{t}, \bar{\lambda}}^\ell \text{ for } \beta < \alpha \text{ we have } X \subseteq \bigcup_{\beta < \alpha} X_\beta \right\}.$$

Also let $J_{\bar{t}, \bar{\lambda}, \theta}^\ell = J_{\bar{t}, \bar{\lambda}, < \theta^+}^\ell$.

3.5 Claim. 1) $J_{\bar{t}, \bar{\lambda}, < \theta_1}^{i(1)} \subseteq J_{\bar{t}, \bar{\lambda}, < \theta_2}^{i(2)}$ when $\theta_1 \leq \theta_2, i(1) \leq i(2)$ are among 4, 5, 6.

2) $J_{\bar{t}, \bar{\lambda}, \theta}^i$ is a θ^+ -complete ideal for $i = 4, 5, 6$.

3) If $\lambda_\ell \geq \beth_{t_\ell-1}(\theta)$ then the ideal $J_{\bar{t}, \bar{\lambda}, \theta}^i$ is proper for $i = 4, 5, 6$.

Proof. 1), 2) Easy.

3) As in 2.4. □_{3.5}

3.6 Definition. For $\bar{\lambda} = \langle \lambda_\ell : \ell < \omega \rangle, \bar{t} = \langle t_\ell : \ell < \omega \rangle$ such that $2 \leq t_\ell < \omega$ we define

- (A) $B_{\bar{t}, \bar{\lambda}}^{\text{rtf}}$ is the free (abelian) group generated by $\{x_\alpha^m : m < \omega, \alpha < \lambda_m\}$.
- (B) Let $B_{\bar{t}, \bar{\lambda}, n}^{\text{rtf}}$ be the subgroup of $B_{\bar{t}, \bar{\lambda}}^{\text{rtf}}$ generated by $\{x_\alpha^m : m < n \text{ and } \alpha < \lambda_m\}$
- (C) $G_{\bar{t}, \bar{\lambda}}^{\text{rtf}}$ is the pure closure in $(B_{\bar{t}, \bar{\lambda}}^{\text{rtf}})^{[\bar{t}]}$ of the subgroup of $(B_{\bar{t}, \bar{\lambda}}^{\text{rtf}})^{[\bar{t}]}$ generated by

$$B_{\bar{t}, \bar{\lambda}}^{\text{rtf}} \cup \left\{ \sum_{m < \omega} \left(\prod_{\ell < m} t_\ell \right) (x_{(\eta(\ell))(1)}^\ell - x_{(\eta(\ell))(0)}^\ell) : \eta \in \prod_{\ell < \omega} [\lambda_\ell]^2 \right\}$$

(here we use that if e.g. $\eta(\ell) = \{\alpha, \beta\}, \alpha < \beta$ then $(\eta(\ell))(1) = \beta, (\eta(\ell))(0) = \alpha$).

- (D) Let $\bar{B}_{\bar{t}, \bar{\lambda}}^{\text{rtf}} = \langle B_{\bar{t}, \bar{\lambda}, n}^{\text{rtf}} : n < \omega \rangle$.

3.7 Definition. Assume

$\boxtimes_{H,\bar{H}}^{\bar{t}} \bar{H} = \langle H_n : n < \omega \rangle$ is an increasing sequence of abelian subgroups of H ,
such that $\bigcup_{n < \omega} H_n$ is dense in H by the \bar{t} -adic topology.

Then we let

$$I_{H,\bar{H}}^{4,\bar{t}} = \left\{ X \subseteq H : \text{for some } n < \omega, \text{ the intersection of the } \bar{t}\text{-adic closure of } PC(X)_H \text{ in } H, \right. \\ \left. cl_{\bar{t}}(PC(X)_H, H) \text{ with } \bigcup_{\ell < \omega} H_\ell \text{ is a subset of } H_n \right\}$$

$$I_{H,\bar{H},<\theta}^{4,\bar{t}} = \left\{ X \subseteq H : \text{for some } \alpha < \theta \text{ and } X_\beta \in I_{H,\bar{H}}^{4,\bar{t}} \text{ for } \beta < \alpha \text{ we have } X \subseteq \bigcup_{\beta < \alpha} X_\beta \right\}$$

$$I_{H,\bar{H},\theta}^{4,\bar{t}} = I_{H,\bar{H},<\theta^+}^{\bar{t}}.$$

3.8 Definition. Assume $\bar{t} = \langle t_\ell : \ell < \omega \rangle, 2 \leq t_\ell < \omega$, and

$\boxtimes_{H,\bar{H}}^{\bar{t}} H$ is Hausdorff in the $\bar{t} \upharpoonright [k, \omega)$ -topology for each $k < \omega$ where $t \upharpoonright [k, \omega) = \langle t_{k+\ell} : \ell < \omega \rangle$. Further $\bar{H} = \langle H_n : n < \omega \rangle$ is an increasing sequence of abelian groups, $\bigcup_{n < \omega} H_n \subseteq H$ is dense in the $\bar{t} \upharpoonright [k, \omega)$ -adic topology for each $k < \omega$.

Then we let

$$1) \quad I_{H,\bar{H}}^{5,\bar{t}} = \left\{ X \subseteq H : \text{for some } n(*) < \omega, \text{ for every } n \in (n(*), \omega) \text{ there is no} \right. \\ \left. y \in H_{n+1} \text{ such that: } d_{\bar{t} \upharpoonright [n, \omega)}(y, PC(\langle X \rangle)) = 0 \text{ but } d_{\bar{t} \upharpoonright [n, \omega)}(y, H_n) > 0 \right\}$$

$$I_{H,\bar{H},<\theta}^{5,\bar{t}} = \left\{ X : \text{there are } \alpha < \theta \text{ and } X_\beta \in I_{H,\bar{H}}^{5,\bar{t}} \text{ for } \beta < \alpha \text{ such that } X \subseteq \bigcup_{\beta < \alpha} X_\beta \right\}.$$

$$\text{Moreover } I_{H,\bar{H},\theta}^{5,\bar{t}} = I_{H,\bar{H},<\theta^+}^{5,t}.$$

$$2) \quad I_{H,\bar{H}}^{6,\bar{t}} \text{ (and } I_{H,\bar{H},<\theta}^{6,\bar{t}}, I_{H,\bar{H},\theta}^{6,\bar{t}}) \text{ are defined similarly except that we demand } n(*) = 0.$$

$$3) \quad I_{\bar{t},\bar{\lambda}}^{i,\text{rtf}} \text{ means } I_{G_{\bar{t},\bar{\lambda}}^{\text{rtf}}, \bar{B}_{\bar{t},\bar{\lambda}}^{\text{rtf}}}^{i,\bar{t}} \text{ where } \bar{B}_{\bar{t},\bar{\lambda}}^{\text{rtf}} = \langle B_{\bar{t},\bar{\lambda},n}^{\text{rtf}} : n < \omega \rangle.$$

3.9 Claim. For $\bar{\lambda}, \bar{t}$ as in 3.4

- (a) we have $\boxtimes_{G_{\bar{t}, \bar{\lambda}}^{\text{rtf}}, \bar{B}_{\bar{t}, \bar{\lambda}}^{\text{rtf}}}$ (from 3.8)
- (b) $G_{\bar{t}, \bar{\lambda}}^{\text{rtf}}$ is \aleph_1 -free; moreover $G_{\bar{t}, \bar{\lambda}}^{\text{rtf}}/B_{\bar{t}, \bar{\lambda}, n}^{\text{rtf}}$ is \aleph_1 -free for each $n < \omega$
- (c) $I_{\bar{t}, \bar{\lambda}, \theta}^{i, \text{rtf}}$ are θ^+ -complete ideals for $i = 4, 5, 6$
- (d) if $\boxtimes_{H, \bar{H}}^{\bar{t}}$ (from 3.8) and $i \in \{4, 5, 6\}$ then $I_{H, \bar{H}, \theta}^{i, \bar{t}}$ is a θ^+ -complete ideal.

Proof. Straightforward (for (6), use an argument similar to that of 3.3).

The following lemma connects the combinatorial ideals defined above and the more algebraic ideals defined in 3.7.

3.10 Claim. 1) Assume

- $\boxtimes_1 \bar{t} = \langle t_\ell : \ell < \omega \rangle, 2 \leq t_\ell < \omega$
- $\boxtimes_2 \bar{\lambda} = \langle \lambda_\ell : \ell < \omega \rangle$, and $\lambda_\ell > \beth_1(\theta)$ for $\ell < \omega$.

Then the ideal $I_{\bar{t}, \bar{\lambda}, \theta}^{i, \text{rtf}}$ is proper for $i = 4, 5, 6$.

2) Assume \boxtimes_1 and

- $\boxtimes'_2 \bar{\lambda} = \langle \lambda_\ell : \ell < \omega \rangle, \lambda_\ell = \aleph_0, \theta = \aleph_0$.

Then the ideal $I_{\bar{t}, \bar{\lambda}, \theta}^{i, \text{rtf}}$ is proper.

Proof. 1) If not, we can find $X_\alpha \subseteq L =: G_{\bar{t}, \bar{\lambda}}^{\text{rtf}}$ for $\alpha < \theta$ such that $G_{\bar{t}, \bar{\lambda}}^{\text{rtf}} = \bigcup_{\alpha < \theta} X_\alpha$

and $X_\alpha \in I_{\bar{t}, \bar{\lambda}}^{i, \text{rtf}}$. For $\alpha \leq \omega$ and $\eta \in \prod_{\ell < \alpha} [\lambda_\ell]^2$ we let

$$x_\eta = \sum_{m < \alpha} \left(\prod_{\ell < m} t_\ell \right) (x_{(\eta(n))(1)}^m - x_{(\eta(n))(0)}^m).$$

As in the proof of 2.4, we apply [RuSh 117], [Sh:f, Ch.XI, 3.5] for the ideal $J_\ell = ERI_\theta^2(\lambda_\ell)$ (this ideal is, of course, θ^+ -complete and non-trivial as $\lambda_\ell > 2^\theta$).

So we can find $\langle Y_m : m < \omega \rangle, \langle A_\eta : \eta \in Y_m \rangle$ and $\alpha(*) < \theta$ such that

- (a) $Y_m \subseteq \prod_{\ell < m} [\lambda_\ell]^{t_\ell}$
- (b) Y_0 is a singleton
- (c) $A_\eta \in (J_{\ell g(\eta)})^+$ for $\eta \in Y_m$ (so $A_\eta \subseteq [\lambda_{\ell g(\eta)}]^{t_{\ell g(\eta)}}$)
- (d) $Y_{m+1} = \{\eta \hat{\ } \langle u \rangle : u \in A_\eta, \eta \in Y_m\}$
- (e) if $\eta \in Y_m$ then $\eta \in \{\nu \restriction m : x_\nu \in X_{\alpha(*)}\}$.

Note that we can demand that for all $\alpha < \beta$ from $A_\eta, d(x_\beta^{\ell g(\eta)} - x_\alpha^{\ell g(\eta)}, \langle X_{\alpha(*)} \rangle)$ is the same. We now prove by induction on $k < \omega$ that

- (*)_k for any $m < \omega$, if $\eta \in Y_m$ and $A \subseteq A_\eta$ is infinite then for some infinite $A' \subseteq A$ for any $\alpha < \beta$ from A' we have

$$\left(\prod_{\ell < m} t_\ell \right) (x_\beta^m - x_\alpha^m) \in \text{cl}_{\bar{t}}(\langle X_{\alpha(*)} \rangle, L) + \left(\prod_{\ell < m+k} t_\ell \right) L.$$

For $k = 0$ this is trivial: the element $(\prod_{\ell < m} t_\ell)(x_\beta^m - x_\alpha^m)$ belongs to $(\prod_{\ell < m+k} t_\ell)L$.

For $k + 1$, by $(*)_k$ as we can replace every A_η by an infinite subset without loss of generality for $m < \omega$, A_η can serve as A' in $(*)_k$. Let $\eta \in Y_m$, and let $\gamma_0 < \gamma_1 < \gamma_2 < \dots$ be in A_η . So for each $j < \omega$, let $\eta_j \in Y_{m+k+1}$ be such that $\eta_j \upharpoonright m = \eta$, $\eta_j(m) = \{\gamma_j, \gamma_{j+1}\}$. By clause (e) above we know that there are ν_j such that $\eta_j \triangleleft \nu_j \in \prod_{\ell < \omega} [\lambda_\ell]^2$ and

(i) $x_{\nu_j} \in X_{\alpha(*)}$.

Now by the definitions of x_{η_j}, x_{ν_j}

(ii) $x_{\eta_j} = x_{\nu_j} \bmod (\prod_{\ell < m+k+1} t_\ell)L$

(iii) if $\ell \in [m+1, m+k+1)$ and $j < \omega$ then

$$x_{\eta_j \upharpoonright (\ell+1)} - x_{\eta_j \upharpoonright \ell} \in c\ell_{\bar{t}}(\langle X_{\alpha(*)} \rangle, L) + (\prod_{i < \ell+k} t_i)L \subseteq c\ell_{\bar{t}}(\langle X_{\alpha(*)} \rangle, L) + (\prod_{i < m+k+1} t_i)L$$

[why? by the induction hypothesis the difference is

$$(\prod_{i < m+\ell} t_i)(x_{(\eta_j(\ell))(1)}^\ell - x_{(\eta_j(\ell))(0)}^\ell)]$$

(iv) $x_{\eta_j} - x_{\eta_j \upharpoonright (m+1)} \in c\ell_{\bar{t}}(\langle X_{\alpha(*)} \rangle, L) + (\prod_{i < m+k+1} t_i)L$

[why? use (iii) for $\ell = m+1, \dots, m+k$, noting that $\ell g(\eta_j) = m+k+1$.]

(v) $x_{\eta_j \upharpoonright (m+1)} \in c\ell_{\bar{t}}(\langle X_{\alpha(*)} \rangle, L) + (\prod_{i < m+k+1} t_i)L$

[why? by (i) + (ii) + (iv)]

(vi) $\sum \{x_{\eta_j \upharpoonright (m+1)} : j < \prod_{i < m+k+1} t_i\} \in c\ell_{\bar{t}}(\langle X_{\alpha(*)} \rangle, L) + (\prod_{i < m+k+1} t_i)L$

[why? by (v)]

(vii) $x_{\gamma_{j(*)}}^m - x_{\gamma_0}^m \in c\ell_{\bar{t}}(\langle X_{\alpha(*)} \rangle, L) + (\prod_{i < m+k+1} t_i)L$ for $j(*) = \prod_{i < m+k+1} t_i$

[why? by (vi) because

$$\begin{aligned} & \sum \{x_{\eta_j \upharpoonright (m+1)} : j < \prod_{i < m+k+1} t_i\} \\ &= \sum \{x_{\eta_j \upharpoonright m} + (\prod_{i < m+1} t_i)(x_{\gamma_{j+1}}^m - x_{\gamma_j}^m) : j < \prod_{i < m+k+1} t_i\} \\ &= \sum \{x_{\eta_j \upharpoonright m} : j < \prod_{i < m+k+1} t_i\} + (\prod_{i < m+1} t_i) \sum \{(x_{\gamma_{j+1}}^m - x_{\gamma_j}^m) : j < \prod_{i < m+k+1} t_i\} \\ & \quad [\text{as } \eta_j \upharpoonright m \text{ does not depend on } j \text{ and obvious arithmetic}] \\ &= (\prod_{i < m+k+1} t_i) \cdot x_{\eta_{j(*)}} + (\prod_{i < m+1} t_i)(x_{\gamma_{j(*)}}^m - x_{\gamma_0}^m) \in \\ & \quad (\prod_{i < m+1} t_i)(x_{\gamma_{j(*)}}^m - x_{\gamma_0}^m) + (\prod_{i < m+k+1} t_i)L \end{aligned}$$

- (viii) if $\rho \in Y_m$ and $\alpha < \beta$ are in A_η then
 $(\prod_{i < m+1} t_i)(x_\beta^m - x_\alpha^m) \in \text{cl}_{\bar{t}}(\langle X_{\alpha(*)}, L \rangle) + (\prod_{i < m+k+1} t_i)L$
 [why? by (vii) and the choice of the $Y_m, A_\eta(\eta \in Y_m, m < \omega)$.]

So we have carried the induction on k .

2) Easier.

□_{3.10}

3.11 Claim. *Assume*

- ⊠₁ $\bar{t} = \langle t_\ell : \ell < \omega \rangle$ and $2 \leq t_\ell < \omega$
- ⊠₂ $\lambda_\ell > \beth_1(\theta)$
- ⊠₃ $\text{cov}(\lambda, (\prod_{\ell < \omega} \lambda_\ell)^+, (\prod_{\ell < \omega} \lambda_\ell)^+, \theta^+) \leq \lambda$.

Then $\mathbf{U}_{J_{\bar{t}, \bar{\lambda}, \theta}^6}(\lambda) = \lambda$ and $\mathbf{U}_{I_{\bar{t}, \bar{\lambda}, \theta}^6}(\lambda) = \lambda$.

3.12 Conclusion. For every $\lambda \geq \beth_\omega$ for some $\theta < \beth_\omega$, for every $\kappa \in (\beth_1(\theta), \beth_\omega)$ for every $\lambda_n \in [\beth_\omega(\theta), \kappa]$ we have

$$\mathbf{U}_{I_{\bar{t}, \bar{\lambda}, \theta}^6}(\lambda) = \lambda = \mathbf{U}_{J_{\bar{t}, \bar{\lambda}, \theta}^6}(\lambda).$$

Proof. By the previous claim and [Sh 460] (similar to 2.6).

□_{3.12}

3.13 Claim. *Assume*

- (a) $\prod_{\ell < \omega} \lambda_\ell < \mu < \lambda = \text{cf}(\lambda) \leq \lambda' \leq \lambda'' < \mu^{\aleph_0}$
- (b) $\mu^+ < \lambda$ or at least for some \mathcal{P} ,
 $(*)_{\mathcal{P}} \quad |\mathcal{P}| = \lambda$ and $(\forall a \in \mathcal{P})(a \subseteq \lambda \ \& \ \text{otp}(a) = \mu)$
 and $(\forall E)(E \text{ a club of } \lambda \rightarrow (\exists a \in \mathcal{P})(a \subseteq E))$
- (c) $\lambda'' = \mathbf{U}_{I_{\bar{t}, \bar{\lambda}}^6}(\lambda') < \mu^{\aleph_0}$ where $t_m = \prod_{\ell < m} \ell!$ or at least $\lambda'' = \mathbf{U}_{J_{\bar{\lambda}}^1}(\lambda')$
- (d) $\text{cov}(\lambda'', \lambda^+, \lambda^+, \lambda) < \mu^{\aleph_0}$ or at least $\mathbf{U}_{\text{id}^a(\mathcal{P})}(\lambda'') < \mu^{\aleph_0}$ where \mathcal{P} satisfies $(*)_{\mathcal{P}}$.

Then we can find \aleph_1 -free abelian groups G_α of cardinality λ for $\alpha < \mu^{\aleph_0}$ such that for every \aleph_1 -free abelian group G of cardinality λ or just $G \in \mathfrak{K}_\lambda^{\text{rtf}}$ we have:

some G_α is not embeddable into G ; also the number of ordinals $\alpha < \mu^{\aleph_0}$ for which G_α is embeddable into G is at most $\text{cov}(\lambda'', \lambda^+, \lambda^+, \lambda)$ (or $\leq \mathbf{U}_{\text{id}^a(\mathcal{P})}(\lambda'')$ at least)

Proof. Like 2.8, not that “ \aleph_1 -free” implies $\| - \|_{\bar{t}}$ is a norm.

3.14 Conclusion. If $\beth_\omega \leq \mu^+ < \lambda = \text{cf}(\lambda) < \lambda^{\aleph_0}$ then in $\mathfrak{K}_\lambda^{\text{rtf}}$ there is no member universal even just for $\mathfrak{K}_\lambda^{\aleph_1\text{-free}}$.

Proof. Straightforward.

3.15 Concluding Remarks. 1) We can in 3.11 - 3.14 use 3.10(2) instead of 3.10(1).
2) In §2 we can use the parallel of 3.10. Also we can deal with the class of R -modules or some natural subclasses of it, we hope to return this elsewhere.

3.16 Remark. If $\lambda = \aleph_0$ there is no universal member in $\mathfrak{K}_\lambda^{\text{rtf}}$. In fact for any $\mathbf{Q} \subseteq \mathbf{P}^*$ let $G_{\mathbf{Q}}$ be the subgroup of $\mathbb{Q}x \oplus \bigoplus_p \{\mathbb{Q}x_p : p \in \mathbf{P}^* \setminus \mathbf{Q}\}$ generated by

$$\begin{aligned} & \{p^{-n}x : p \in \mathbf{Q}\} \cup \{q^{-n}x_p : p \in \mathbf{P}^* \setminus \mathbf{Q} \text{ and } n < \omega, \text{ and } q \in \mathbf{P}^* \setminus \{p\}\} \\ & \cup \{p^{-n}(x - x_p) : n < \omega \text{ and } p \in \mathbf{Q}\}. \end{aligned}$$

So $G_{\mathbf{Q}} \in \mathfrak{K}_{\aleph_0}^{\text{rtf}}$, and if h embeds $G_{\mathbf{Q}}$ into $G \in K^{\text{trf}}$ then $\mathbf{P}(h(x), G) = \mathbf{Q}$.

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